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Exploring informal mathematical products of low achievers at the secondary school level

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Abstract

This article examines the notion of informal mathematical products, in the specific context of teaching mathematics to low achieving students at the secondary school level. The complex and relative nature of this notion is illustrated and some of its characteristics are suggested. These include the use of ad-hoc strategies, mental calculations, idiosyncratic ideas, everyday rather than mathematical language, non-symbolic explanations, visual justifications and common-sense based reasoning. The main argument raised in the article concerns the challenge of valuing informal mathematical products, created by low achievers, and using them within the mathematics classroom as means for advancing such students. The data draws from several research and design projects conducted in Israel since 1991. Selected examples of students' products, gathered from low-track mathematics classrooms involved in these projects, are presented and analyzed. The analyses highlight various features of such products, and portray the possible gains of teaching approaches that legitimize, and build onwards from, informal products of low achievers.

Keywords: Low achievers; Secondary school mathematics; Informal reasoning

During the last decade, research on success and failure in mathematics has undergone quite a considerable shift in its main focuses. The extensive literature on students' misconceptions, errors and alternative ideas, established mainly around the eighties, has put much emphasis on what might be viewed as 'cognitive obstacles' to success in mathematics. Recent studies attempt to widen the arena of investigations beyond documenting and analyzing difficulties in regard to specific mathematical contents, and seek explanations for the phenomenon of unsuccessful mathematics students through a wide spectrum of factors, found in and out of school. Among the central elements associated in contemporary research with differential achievement in mathematics we find (a) students' background factors, such as social class, culture and community (see, for instance, Cooper & Dunne, 2000; Martin, 2000; Moses & Cobb, 2001; Secada, 1992; Tate, 1997; Zevenbergen, 2000), (b) class settings (e.g. ability grouping) and consequent students' personal trajectories (Boaler, William, & Brown, 2000; Chazan, 2000; Zevenbergen, 2003), and (c) teachers' beliefs about the learning potential of different students and about appropriate teaching models for weaker vs. stronger students (Allexsaht-Snider & Hart, 2001; Love, 2002; Raudenbush, Rowan, & Cheong, 1993; Yair, 1997; Zohar, Degani, & Vaaknin, 2001).

This article focuses on yet another potential source for success and failure in mathematics. One which may be expressed through the following question: To what degree are students' informal mathematical products appreciated, valued and adequately treated within the mathematics classroom? In order to discuss this question, some terminology clarifications are needed first. The expression 'informal mathematical products' may elicit different interpretations,

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nuances of meaning, and associations for different readers. Moreover, the term 'informal' by itself is used in diverse ways among researchers in mathematics education, and its meaning often strongly depends on the context in which it appears. Thus, we begin by describing what counts as 'informal mathematical products' in the context of this article. Since we relate specifically to products of low achievers in secondary school mathematics, some descriptive characteristics of low achieving students will be included in this account. Following a brief overview of the data sources, the central part of the article will present a spectrum of examples of informal mathematical products, created by low achievers. Although each example portrays a different element to concentrate upon, the whole picture is, in our view, greater than the sum of these elements. We suggest that all examples share an important feature: They indicate that, when listened to, secondary school low achievers can generate mathematical products that carry a genuine mathematical value. Furthermore, in certain cases the extent and profundity of such products may grow, given a suitable learning environment.

1. Informal mathematical products

By a "mathematical product" we refer to any outcome — in the wide sense of this word — of actions that involve traceable mathematical reasoning. A mathematical product may be expressed through either written, spoken or other illustrated forms, may include words, symbols, diagrams, tables, pictures, solid models or even gestures (as in, for instance, indicating the shape of a curve using one's hands). It can be created as a result of a spontaneous, unguided inquiry (e.g., when a child expresses the revelation that there is "no end" to numbers), it may come as a reaction within a more structured situation such as an exam question or a classroom task, it may be prompted by a motivation to establish new mathematical knowledge, as in the work of professional mathematicians, or it may emerge from various other kinds of social as well as inner triggers.

Taking this broadly defined notion of mathematical products as a starting point, we now turn to elucidate what is intended by "informal mathematical product". We are well aware of the problematic nature of such an attempt: First, the notions of "formal" and "informal" do not constitute a dichotomy, but rather a continuity. As Selden and Selden (1995) note in their discussion of formal and informal statements, informality is "a matter of degree, with some statements more informal than others" (p. 127). We suggest that this holds not only for statements but for mathematical products in general, and thus the identification of a mathematical product as informal is basically a relative one. Moreover, it should be recognized that informality is also relative in regard to different potential producers of a certain product. The degree of informality we attribute to a product may be strongly affected by the mathematical level we expect from its creator. For example, if a primary school student says that an isosceles triangle has two equal angles because it is symmetric, this explanation may count as a formal mathematical product at this stage, as it makes a correct use of designated terms drawn from the mathematical environment of elementary school classrooms. The same statement, made by a high school student studying a course in Euclidean geometry, will probably be regarded as an informal observational mathematical product, while the formal product, i.e., the proof of this geometrical fact, should include the use of relevant theorems. In the same manner, what counts for a mathematician as an informal product, might be considered in secondary school mathematics as a formal one.

Second, in correspondence with what Forman (1996) and Moschkovich (2003) have pointed out with respect to the distinction between everyday and mathematical discourses, it may be difficult to distinguish and separate between formal and informal elements in students' mathematical products.

In spite of these apparent complexities involved in attempting to define informal mathematical products in the general sense, we nevertheless find it feasible to describe and refer to such products within the course of the discussion to follow. We will do so by (a) limiting the discussion to a specific context, concerning low achievers in secondary school mathematics (the term "low achievers" will be addressed in detail later on), and (b) using general distinctions, made in the research literature, that can be 'borrowed' in order to characterize informal mathematical products in this particular context. Referring to secondary school mathematics, at certain grades, limits the expected mathematical level to a relatively narrow spectrum, presumably shared among readers.

Let us begin with an illustrating example. Consider the following question in analytic geometry:

Given the points B(6,3) and C(3,0)

(a) Find a point A on the *y*-axis, so that the angle ACB will be a right angle.
(b) Find the equation of the line through AC.

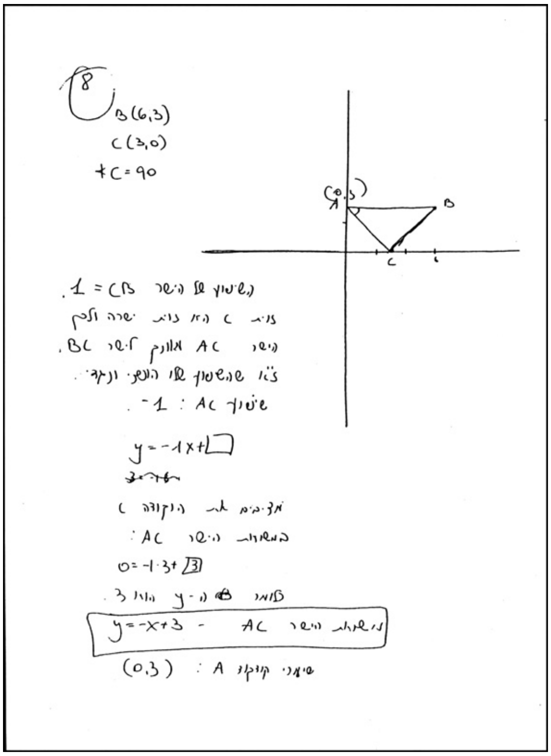

Fig. 1. A student's answer to a problem in analytic geometry (I).

The answer presented in Fig. 1 was given by a student in a low-track mathematics class.

The text written in Hebrew below the axes-system, translated to English, reads:

The slope of the line CB = 1.
Angle C is a right angle and so the line AC is perpendicular to the line BC.
This means that its slope is inverse and opposite.
The slope of AC: −1
y = - 1 x + $\boxed{}$
You put the point C in the equation of the line AC:
0 = - 1 · 3 + $\boxed{3}$
That is, the *y* is 3.

The equation of the line AC- y = -x + 3

The values of vertex A: (0,3)

What components of this successful answer can be regarded as "informal"? First, it can be seen that the student found the slope of BC with no written calculations. It can therefore be presumed that she had used visual means, in other words she read the slope directly from the graph she drew. Second, the student's argument is more narrative than symbolic in nature. Avoiding formal mathematical notations such as the formula $m_1 \cdot m_2 = -1$, the student instead applied considerations written in words (using accurate terms as perpendicular, inverse, and opposite), to obtain the value of −1 for the slope of the line AC. Third, the strategy used to obtain the *y*-intercept value of this line (commonly marked as *n* in the equation $y = mx + n$) was not the fully formal strategy. The unknown value was represented as a box to be filled with a number that correctly 'fits' the numerical statement.[1] The student did not refer to the symbol *n* even

[1] This strategy was introduced to the student and her classmates during instruction, given as part of the '3U' Curriculum Project. Details about this project appear later in the article.

after the value of 3 was found. Instead, the remark "That is, the *y* is three" suggests that she was aiming at finding the *y*-intercept. Lastly, it is notable that the student first solved section (b) of the question, and then used it to obtain the answer for section (a). This way she had avoided the equation $(y_A - 0)/(0 - 3) = -1$, which is the standard formal way to solve section (a) directly. In sum, this answer reflects the use of visualization and alternative common-sense based strategies, along with preference of words over symbols and avoidance of formulas. These characteristics may serve to identify informal mathematical products, and we shall refer to them, among others, in the literature review below.

1.1. Characterizing informal mathematical products in light of existing research

Several well-known studies have established distinctions between in-school and out-of-school mathematics. For example, Lave (1988) and Nunes, Schliemann, and Carraher (1993) show that subjects who successfully performed various calculations with nearly no mistakes when observed in natural settings, erred considerably more when presented with similar calculations in school-like questionnaires. The strategies used in natural situations were mostly ad-hoc strategies, relating to specific features of the given situation, and based on mental calculations and estimations used in a flexible manner. We claim that using this kind of strategies can be associated with creating informal mathematical products, whether these products are created in or out of school.

Another distinction was made more recently by Ben-Yehuda, Lavy, Lynchevski, and Sfard (2005), between colloquial mathematical discourses and literate mathematical discourses. While colloquial mathematical discourses involve the use of everyday language and seem to develop spontaneously as a by-product of repetitive experiences, literate mathematical discourses are characterized as being mediated by symbolic representations created specifically for the purpose of communicating about quantities. We see the former as linked with informal mathematical products, whereas the latter may be associated with formal products.

Leron (2004) differentiates between three levels of mathematics, of which the second is termed 'informal mathematics'.[2] He suggests that informal mathematics is an extension of common sense, which is mainly "carried out with the help of figures, diagrams, analogies from everyday life, generic examples, and students' previous experience" (p. 219). As opposed to that, formal mathematics (the third level) is characterized by Leron through its "full apparatus of abstraction, formal language, de-contextualization, rigor and deduction" (p. 220).

In accordance with the above-mentioned distinctions, we suggest that informal mathematical products in secondary school mathematics may be characterized by the use of one — or a combination of — the following: ad-hoc strategies, mental calculations, idiosyncratic ideas, everyday rather than mathematical language, verbal explanations, visual justifications, common-sense based reasoning, and bypass of formulae and symbolic notations. Examples of such informal mathematical products will be presented throughout the central part of this article.

1.2. Low achievers in mathematics as creators of informal products

The question we wish to focus on is: To what extent are informal mathematical products valued and appreciated within the mathematics classroom, in ways that can advance low achieving students? It seems that customarily, formal symbolic products are viewed as the almost sole desired and valued outcomes of mathematics learning. For example, Morgan (1998) refers to "a common perception among mathematicians that the only significantly meaningful part of a mathematics text resides in the symbol system" (p. 12). According to this perception, "'correct mathematics' may be seen as equivalent to producing a correct sequence of symbols . . ." (p. 12), a view that is shared by many mathematics teachers as well (Tobias, 1989).

Ben-Yehuda et al. (2005) criticize this view, with specific reference to students at risk. In their study, two 17-year-old girls with a history of difficulties and failure in mathematics were interviewed. Of the two, the girl characterized by her teacher as having a lower 'mathematical potential' due to her poor ability to perform formal procedures, was found by the researchers to be more capable of flexibly moving from one mental strategy to another and in general more likely to benefit from appropriate intervention. Ben-Yehuda et al. find unjustifiable "the fact that colloquial mathematical discourses are depreciated, sometimes to the degree of their total banishment from school curricula. [. . .] Such de-legitimization may bar the participation of those students for whom the depreciated forms of communication would

[2] The first level is termed rudimentary arithmetic.

be the most accessible" (p. 220). This critique appears to correspond with an earlier one, made by Turkle and Papert (1992), who had challenged "the hegemony of the abstract, formal, and logical as the privileged canon in scientific thought" (p. 3).

In agreement with these criticisms, we claim that low achieving students can benefit much from teachers' appreciation and legitimization of their informal mathematical products. At this point it should be stressed that we acknowledge the appropriateness of such a policy for *all* students. The need to listen, respect and build on students' informal ideas is already well recognized by mathematics educators (e.g., Arcavi & Isoda, in press; Cooney & Krainer, 1996; Hiebert et al., 1997). However, we argue that for low achieving students this approach could be crucial, to the degree of making the difference between success and failure.

Research regarding learning and thinking characteristics of low achievers in mathematics show that students, who are streamed in secondary schools to low-track settings due to past failures, can bring forward valuable informal ideas (Chazan, 2000; Karsenty & Arcavi, 2003). However, if they constantly feel that their products are not accepted as meaningful, they tend to withdraw altogether from further contributions. In this respect, good practice on the part of teachers includes the capability to pick up an informal product of a student, listen (or read, refer, etc.) to it carefully, detect the sense in it, and convey its value back to the student. At a deeper level, the teacher can *appropriate* the student's product, in the sense described by Moschkovich (2004). That is, the product is taken as a starting point for a subsequent joint productive activity, or a new task, based on the teacher's interpretation of the student's product. The teacher's understanding of the initial product is reflected back to the student and thus serves to create shared "new meanings for words, new perspectives and new goals and actions" (p. 51).

In the following, we explore the nature and potential value of informal mathematical products. We present and analyze selected examples, gathered in the course of several classroom studies and projects in which we were involved in recent years. All the examples to be presented relate to secondary school students who are regarded by their school system as low achievers in mathematics. The term "low achievers", as will appear herein, refers to students who study in regular school settings (as opposed to special education settings), i.e., students who are not defined as having distinct learning disabilities, but rather are considered — mostly due to low scores in tests — as less successful in mathematics than their age group peers. It should also be noted that the data, as shall be specified below, was collected in Israel, where ability grouping in mathematics is common in secondary schools, from eighth grade onwards. We therefore focused on students grouped in lower-track mathematics classes. Although this population of students is not homogeneous in terms of mathematical capabilities, some learning characteristics may be generally recognized in regard to their "academic portrait" (Chazan, 1996, p. 458). Among these characteristics, as pointed out by Chazan (1996, 2000) and Arcavi, Hadas, and Dreyfus (1994), are short lived memory for mathematical procedures, short concentrating periods, difficulties with reading and writing in a mathematical language, poor note-taking and homework habits, and low frustration threshold. At the metacognitive level, it was found that low-achieving students have difficulties in planning, self-monitoring and reflecting on their actions in the course of handling a given mathematical task (Cardlle-Elawar, 1995). Lastly, but especially important, is students' inclination to view mathematics as an esoteric subject, detached from their common sense experience (Karsenty & Arcavi, 2003).

We will not refer in this paper to the important issues of whether and how tracking affects students, or how different settings may or may not alter some of the learning characteristics mentioned, issues which have been widely discussed in recent years (see Boaler, 1997; Boaler et al., 2000; Linchevski & Kutcher, 1998; Oakes, 1990; Zevenbergen, 2003). Rather, in light of the still prevalent occurrence of tracking (at least in Israel), we take this situation as given, and argue that within this setting there is still much to be done.

2. Data sources: an overview of design and methodology of three projects

As implied earlier, the purpose of this article is to bring forward and examine the following argument: Informal mathematical products of low achievers may serve both as indicators of students' capabilities to perform meaningful mathematical reasoning, and as significant resources and springboards for advancement of such students. We attempt to ground this argument in several key examples, observed during three connected projects in which we were engaged since 1991: a curriculum project, a classroom study and an instruction-oriented project, all designed to address issues relating to low achievers in secondary school mathematics. This section presents a short background on each of these projects.

2.1. The '3U' curriculum project

From 1991 onwards, a team at the Science Teaching Department of the Weizmann Institute of Science in Israel has developed a new mathematics curriculum[3] for low-track mathematics students in grades 10–12. The aim of this project was to enable more students to pass the Matriculation Exam in mathematics, which is compulsory for obtaining a secondary school Matriculation Certificate in Israel. Even at the lowest level (three credit points, or 'units', as they are called), a large percentage of students fail the exam, and many more do not even attempt it. Without the Matriculation Certificate, these students are prevented from entering many occupations as well as further studies.

The curriculum project, which was named '3U' (three units), introduced learning materials that were based, in general, on the approach of "learning by doing". Classroom activities were designed to form experiences which build on students' common sense, emphasizing the meaning of concepts, and playing down formal treatments, technical manipulations and heavy notation (Arcavi, 2000; Arcavi et al., 1994). The project had a strong component of "research in action", with the main question being: how can the design of learning materials match students' characteristics, keep them working, engage their common sense, and strengthen their confidence in their abilities to do meaningful mathematics and succeed in high-stakes exams? Three experimental classes were taught by members of the development team, throughout the school years of 1993–1995. During these years, a pilot edition of 3U books was published and circulated (for further details about the design process, see Arcavi, 2000). At the end of the 1995 school year, there were altogether 54 students in the experimental classes who took the Matriculation Exam in mathematics.[4] Out of these 54, 49 have passed the exam, with a mean grade of 78.5 points (out of 100 points; $SD = 14.6$). All of these students were initially tracked by their schools to non-matriculation bound math classes.

Data collection for the 3U project included observations of students' work during mathematical lessons in the experimental classes, and analyses of students' written products. These included the 54 completed forms of the final Matriculation Exam, which students in the experimental classes took at the end of the experiment.[5] Examples 2–4, to be discussed in the main section of this article, are taken from this source. Example 5 is taken from a class who studied with the 3U materials after the pilot edition was published.

2.2. The middle-school classroom study on low-track mathematics

During the years 1999–2003 we have conducted a study (funded by the Israeli Ministry of Education) on mathematics learning in low-track classes in grades 8 and 9. This qualitative study concentrated on six low-track mathematics classes, in six different middle schools. Data collection included (a) observations of whole lessons (b) documentation of individual students' work during lessons, usually in interaction with a researcher, who would sit beside a certain student and ask him/her questions concerning the specific task, and (c) semi-structured individual interviews with students, conducted outside the class. Of the six classes, five were visited for periods of between a month and 2 months. The sixth class (grade 8) was visited once a week throughout the school year.

Examples 1 and 6, presented in the following, are taken from this project.

2.3. The SHLAV Project

SHLAV[6] is an instruction-oriented project which started off in 2004. The rationale underlying this project was that there is, in our view, a pressing need to establish more solid links between theory and practice in regard to the issue of secondary school low achievers in mathematics. The two projects described above proposed suitable learning

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[3] The term 'curriculum' is used here according to the following definition: "Curriculum: a conglomerate of materials of different kinds (textbooks, teacher guides, problem books, activity kits, games, material displays, computer software, teacher resource files, etc.) designed to teach and learn mathematics, which share an implicit or explicit rationale on both (a) the goals and (b) the nature of teaching and learning and of mathematical activity" (Arcavi, 2000, p. 156).

[4] The number of students who began the experimental program was somewhat higher, however during the 3 years of the experiment, few students dropped out and a few more felt confident enough to move into regular matriculation-bound math classes.

[5] The final exam given to the experimental classes at the end of the 3 years was similar to the national exam, but was specially designed to take into account the emphases of the approach.

[6] This Hebrew name means "stage", "phase" or "step" (in a staircase) and results from the acronym for "Improving Mathematics Learning".

materials and some insights about the learning processes of students with difficulties. It therefore made much sense to us to create a unique framework where these inputs could be exploited in order to address the needs of low achievers through direct contact with students and appropriate instruction for teachers. Thus, we created a model for a professional role, defined as ‘a counselor focusing on difficulties and low achievement in mathematics’. This role is carried out in secondary schools by an insider, that is, a person who is part of the school staff. This specialized counselor is responsible for all aspects of school activities concerning low achievers in mathematics. During the school years of 2004–2006, the first author has been practicing this experimental position (preformed during one to two full days a week) within a school, located in an Israeli city known to have a low proportion of students who pass the Matriculation Exam in mathematics. The school’s student population included a large portion of new immigrants and youth from low socio-economic backgrounds. The main activities carried out as part of the counseling included (a) instructing the mathematics teachers on learning materials and teaching strategies that promote participation of low-track students (b) conducting teaching sessions in small groups with students in need and (c) organizing and supervising the work of volunteering tutors (see Karsenty, 2006). As part of the project, the counselor has documented mathematical products of students with which she was in close contact. Some of these products are interwoven in the sections to follow.

3. Analyses of selected examples of informal mathematical products created by low achieving students

In this section we present and analyze examples of informal mathematical products created by secondary school low-track students. The various cases highlight different features of informal mathematical products. In addition, they demonstrate how informal products may serve both as indicators of students’ understanding, and as a basis for further learning and teaching.

3.1. The fragility of communicating informal mathematical products

3.1.1. Example 1: Uri solves an algebra word problem

Uri[7] was an eighth grader in a regional junior high school, included in the Middle-School Classroom Study on Low-Track Mathematics, described above. As most of his peers, Uri came from a middle-class home. In class, Uri had difficulties in concentrating for significant periods of time. Together with about 12 other students (the number varied during the year), he was studying mathematics in the lowest track available for his age group. His class was visited by the first author throughout the 2002–2003 school year. As the teacher commonly handed out worksheets, and then attended students’ needs individually, the researcher had many opportunities to sit next to different students, observe their work, assist them if requested and ask questions about their actions. Occasionally she sat beside Uri. She noticed that when given relatively short and focused tasks, Uri was able to concentrate, think and present a reasonable solution. However, there was an evident gap between this ability as expressed verbally, and the products of his thinking as manifested in writing. The following situation portrays this gap.

Near the beginning of the school year, students were working on introductory problems to linear equations. Preceding formal-symbolic equations, these problems are meant to develop some sense of the notion of an unknown value, which can be found by means of educated guesses and the “working backwards” strategy. The problem below is a typical example:

> I thought of a number, I added 7 to this number, and then I multiplied the result by 3. I got 36. What was the number I thought about?

In general, students did not find this type of problems difficult, presumably because of two main reasons: (a) reversing the chain of arithmetic operations made sense to them, since they recognized that these questions present a way from the unknown to the result, while their aim was to arrive from the result back to the unknown number; (b) the reversed arithmetic chain was relatively easy to carry out mentally or with the aid of a calculator. At this stage, there is no reason to expect that students will feel the need for using symbols. The next stage, therefore, is presenting problems that cannot be “worked backwards” easily, in order to initiate recognition of the necessity for a more advanced strategy, a recognition that will hopefully assist in accepting the algebraic symbols. Thus, during one of the lessons in

[7] A pseudonym, as are all names of subjects appearing in this article.

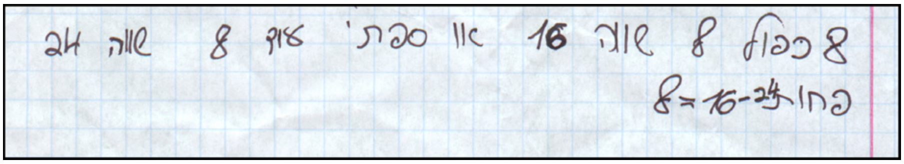


Fig. 2. Uri's written answer as appeared in his notebook.

which students were engaged with simple problems such as the example above, the researcher presented Uri with an additional task, as follows:

> I thought of a number, and I multiplied it by 3. I subtracted 16 from the result, and I ended up with the same number I started with. What was the number?

Within less than a minute, Uri gave the correct answer — the number is 8. His argument, given verbally, was this: "Sixteen is two times eight, out of three times the number". We consider this answer to be a successful informal mathematical product. Uri's argument reflects a simple and reasoned analysis of the given data, based on common sense, communicated through everyday language with no symbolic manipulations involved. However, when requested to write down his argument (not before he was complimented for it), Uri wrote the answer appearing in Fig. 2 (in Hebrew).

It is quite difficult to translate this answer into English in a way that will accurately convey the "spirit" of it, with its spelling and grammar mistakes and missing commas. Still, the following translation may be close:

8 multiply by 8 equals 16 I added another 8 equals 24 subtract 8 = 16–24

This answer reveals genuine difficulties in transferring ideas from a verbal mode to a formal written language. Not only does Uri write 'multiply' when he apparently means 'plus', and '8 = 16–24' instead of '24–16 = 8', but also the flowing, straightforward nature of his original argument seems to be lost. Difficulties in expressing formal numerical ideas in writing are a well-known phenomenon in the mathematical education literature,[8] and may be ascribed to several reasons: cognitive (e.g. a type of dyslexia), behavioral (e.g. lack of suitable practice), or yet another personal problem. However, the question we would like to focus on goes beyond these possible circumstances: Why didn't Uri write his argument *as he said it*, instead of struggling with a formal numeric answer? We suggest that the analysis offered by Ben-Yehuda et al. (2005) supplies one possible explanation for this course of action. As mentioned earlier, Ben-Yehuda et al. refer to the de-legitimization of colloquial mathematical discourses often encountered by students. Such de-legitimization becomes a part of students' low self-evaluation of their mathematical ideas. In other words, students come to depreciate their informal mathematical products. It may be that Uri did not consider a written simple-worded text to be "mathematical enough", and thus sought an answer that will include arithmetic operations and number expressions. If this was indeed the situation, it demonstrates that for students like Uri the de-legitimization of informal products can be critical, since to them more formal mathematical products are much less accessible. However, even if one explains Uri's action differently, the point remains that his informal reasoning was *fragile*, in the sense that it did not 'survive' the transition to a written form. Such fragility may lead to problematic consequences when it comes to conventional assessment procedures. One can assume that had Uri's written answer been given as part of an exam, it would not have obtained a full credit (probably much less than that). On the one hand, this seems unjust in light of Uri's capability to perform a correct — and fairly sophisticated — mental reasoning, as reflected in his verbal answer. On the other hand, the target of communicating mathematical ideas through written texts can hardly be disregarded, and therefore rejecting such an answer does not appear as an unsound action. Uri's case thus emphasizes the need to provide low achievers in mathematics with basic tools of writing, that will assist them in articulating the outcomes of common sense reasoning (which may well be correct and judicious, like identifying 16 as $2x$ in this example), and transferring them into written word sentences. Regardless of the origin of their difficulty, students can gain from an

[8] One of the earlier works on this subject is that of Newman (1977), who included 'encoding' as the fifth category in her model for solving one-step mathematical problems. The term 'encoding', as defined by Newman, refers to students' ability to write their answers in an acceptable form. Newman's error-classification related, among other things, to failures in the encoding stage of the solution.

approach that nurtures and encourages the writing of verbal explanations, as this may contribute to the consolidation of their reasoning. Let us portray this possibility with another short example.

During the pilot phase of the SHLAV Project, we came to know Samuel, an eighth grade student with a history of constant failures in mathematics and a notorious reputation as a troublemaker. During the study of basic linear equations and word problems, Samuel commonly turned in test assignments with final answers only, and no explanations whatsoever about the ways by which he reached those answers. Accordingly, he often received low credits for correct, yet unexplained, answers. His math teacher could hardly uncover his mathematical ideas, neither during class, where she strived to control his undisciplined behavior, nor through homework assignments, which he seldom prepared. However, as the SHLAV counselor discovered, Samuel responded very well to mathematics tutoring in an individual setting, where he behaved in a completely different manner and was attentive and cooperative. He was therefore assigned a volunteer tutor. One of the first actions taken by the tutor was to ask Samuel to explain his answer to a recent test assignment. The question was: "Find two consecutive numbers, so that their sum is 51". At this stage of learning, students were expected to state and solve the equation $x+x+1=51$. Samuel, however, avoided as much as he could using letters for representing unknown values (and he was certainly not unique in that, as we came to realize during our research (Karsenty & Arcavi, 2003)). The verbal account he presented, at the request of the tutor, was as follows: "I did fifty one divided by two, it's twenty five and a half, and then I raised it and I got twenty five and twenty six". This reasoned (though somewhat 'alternative', in terms of language) explanation did not appear in Samuel's written answer, which included only the final numbers 25 and 26. Samuel's mental informal product — in this case an idiosyncratic product that did not evolve as a result of instruction — was therefore hidden by deficient communication. Moreover, the fragility of this product is salient: It is not at all clear if Samuel was fully aware of the generality of his proposed method. It may be that, like many other low achieving students, he came up with an ad-hoc strategy to solve the problem at hand, yet without being able to fully articulate it into a clear argument that can be generalized.

Once again, we argue that students like Samuel should be encouraged to write down their non-symbolic explanations, using simple words and numbers to express their way of thinking. Such practice may lead the student to explicate ideas, making them clearer to oneself and to others. However, regardless of whether this indeed occurs, the act of promoting informal writing conveys by itself an important message to students, i.e., that their ways of thinking are recognized as significant, and therefore it is worthwhile to try to articulate them. Such a message reflects a general perspective that views formal-symbolic writing as just one of several possible means of communicating mathematical ideas. This perspective is compatible with the one offered by Turkle and Papert (1992) in their discussion about epistemological pluralism in science (with specific reference to computer science). Using the terms 'hard' and 'soft' approaches to programming, Turkle and Papert claim that ascribing intellectual value to soft ways of thinking "undermines the elitist position of the 'hards' [. . .], it legitimates alternative methods" (p. 31), and therefore, in the larger intellectual culture, challenges "the superiority of algorithmic and formal thinking" (p. 31).

From a more practical perspective, encouragement of informal writing can potentially enable low achievers to "taste success", as they learn to express their reasoning and gain credit for it. This, in fact, was one of the guiding principles of the 3U Curriculum Project previously described. The example introduced in the next section, drawn from the data of this project, further illustrates this point, while adding yet another element to the discussion.

3.2. Bypassing formulae with sound informal mathematical products

The assumption underlying the 3U learning materials for non-mathematically oriented high school students was that these students are able to construct meaningful mathematical knowledge, based more on common sense and informal reasoning than on formal-symbolic treatments. Hence, teaching according to these materials encourages and practices the use of written mathematical products that play down formal procedures and formulas in favor of word explanations, pictures and sketches. The following example demonstrates the possibility to handle a mathematical task of a 'classic' type, using verbal explanations and graphical sketches instead of symbolic manipulations. The example is taken from the final exam submitted by one of the students in the experimental classes of the 3U project.[9]

[9] This example is also cited in Arcavi (2003), where it is introduced in a context of visual representations.

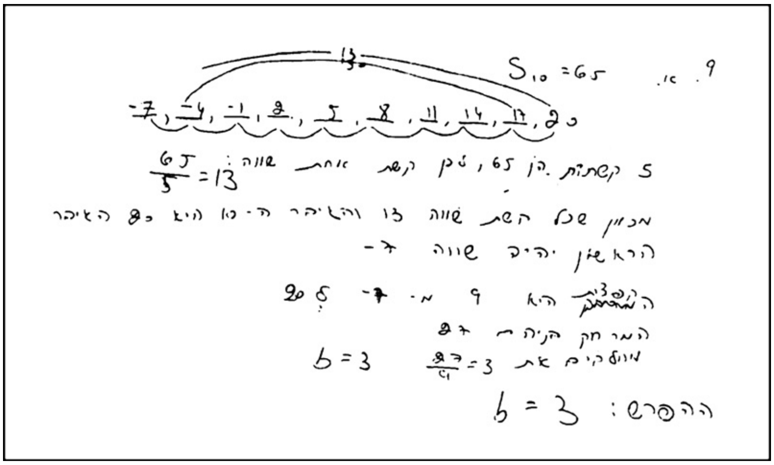


Fig. 3. A student's solution to an arithmetic sequence problem (I).

3.2.1. Example 2: a student's solution to an arithmetic sequence problem

Consider the following problem:

In an arithmetic sequence, it is given that the tenth term equals 20, and the sum of the first 10 terms equals 65. Find the first term and the difference of this sequence.

Usually, the expected solution to this problem involves constructing and solving a system of two equations, on the basis of known formulae, as follows:

$$a_1 + 9d = 20$$
$$10a_1 + 45d = 65$$

For students with difficulties in formal mathematical language, this procedure is not only foreign to their common sense but also error prone due to either inaccurate recall of the formula, incorrect substitution of numbers or technical flaws in solving equations. Let us now examine the solution appearing in Fig. 3.

The text written after the first two lines, translated to English, reads:

5 arcs [refers to the large arcs above the numbers] are 65, so one arc equals: 65/5 = 13
Since each arc equals 13 and the tenth term is 20 the first term will be equal to −7
The jumps [refers to the small arcs beneath the numbers] is 9 [sic] from −7 to 20
The distance between them is 27
Divide the 27/9 = 3 $b = 3$
The difference: $b = 3$

In this solution, the student relied on a simple property of arithmetic sequences, explained in the 3U materials: *There is an equal sum to all pairs of two symmetrically situated elements in the sequence.* This principle makes sense to learners once the basic notion of a constant difference is introduced, and is easily visualized as large arcs, connecting pairs of symmetrically situated elements. The sum of the five arcs adds up to the given sum of 65, and therefore each arc "equals" 13 (this calculation was probably performed with a calculator). Using the widest arc and the given last term yields the required value of the first term, −7 (the calculation 13–20 might have also been done with a calculator). The second intuitive element used by the student is the idea of "jumps", i.e., how many times the difference is added while going from the beginning of the sequence to its end, sketched as small arcs under the blank spaces (which were eventually filled with numbers). There are nine jumps covering a distance of 27 (from −7 to 20), so each jump (that is, the required difference) equals 3.

Thus, the student has managed to solve the task and to fully explain the solution strategy, using sound informal means, without applying formulae or solving equations. The gain of this approach, from the student's point of view, is apparent: A verbal explanation, combined with few simple sketches and calculations, contains much less stumbling blocks than the standard formal method. Yet, another important point that should be emphasized here is that the informal strategy used in the answer has also the advantage of preserving — throughout the solution — the *meaning* of the mathematical concepts at the core of the task. In contrast, the symbolic manipulation commonly used for solving such questions, involves a syntax that is not only difficult for students, but also obscures the original meaning of the context; once the equations are derived, the solution becomes detached from the situation of an arithmetic sequence. Hence, choosing an informal approach can, apart from sparing the students some major difficulties, enable them to stay close to the meaning of the concept.

3.3. Visualization as a crucial component of informal mathematical products

In modern mathematics, visualization is conventionally associated with informal reasoning. Until recent years, visual means for solving mathematical tasks were looked upon mostly as a step towards constructing a formal solution. As Cunningham (1991) notes, "The remarkable success of symbolic, formal mathematics in the late 19th and early 20th centuries left mathematics almost totally committed to symbolic work and tended to discredit visual approaches to mathematics" (p. 70).

Barwise and Etchemendy (1991) claim that "despite the obvious importance of visual images in human cognitive activities, visual representation remains a second-class citizen in both the theory and practice of mathematics" (p. 9). However, as Zimmermann and Cunningham (1991) have stated, recent perspectives on mathematics teaching and learning tend to adopt a more balanced view, which takes into account visual dimensions of mathematics. Visualization is receiving more and more recognition as a powerful tool for enhancing students' understanding of mathematical ideas. Arcavi (2003) contends that, among other advantages, visual solutions enable students to engage with concepts and meanings which may be opaque when symbolic solutions are used.

In the context of teaching low achievers in mathematics, we argue that the role of visualization should be enhanced. Sometimes, visual justifications are the only source of success for students. The next case illustrates this phenomenon.

3.3.1. Example 3: a student's solution to a problem in analytic geometry

Presented in Fig. 4 is a student's answer to the following question:

The points $(-1, 10)$ and $(19, -30)$ are located on a given line.

(a) Find the slope of the line.
(b) Find the equation of the line.

As can be seen, the answer consists of three parts.[10] The squared paper shows the student's sketch, the upper circled segment includes a repetition (in Hebrew) of the data and of question (a), with an additional line of calculations, and the lower circled segment includes a repetition of question (b), with the correct equation ($y=-2x+8$) appearing at the bottom. Examining these three parts, one major question is: how did the student arrive at the equation? A close look at the student's product as a whole, strongly suggests that both the slope and the y-intercept value were found by means of the graphical representation. In spite of the relatively high values of the given points, the student has marked them on an axes-system with a scale of one unit per one square side, and then drew the straight line through the marked points. This enabled the accurate reading of the slope as -2 (a decline of two units per a one-unit step to the right). The inscription "$y=-2x+$" reflects this reading and the step-by-step construction of the equation. The y-intercept value of 8 was probably read from the graph, since no calculations concerning this value appear in the answer. Yet, attention should be drawn to the calculation appearing at the right-hand side of the upper-circled segment. It seems to us that this calculation reveals an attempt to justify the result already obtained for the slope, i.e., the value of -2, by a more formal argument. From the sketch it is clear that the vertical and horizontal distances between the two points were correctly found to be 40 and 20, respectively (possibly by counting units). These distances can be used — as is commonly done in the 3U materials — to obtain the absolute value of the slope, through its definition as 'rise (or drop) over run' (the sign of the slope is separately determined by looking at the direction

[10] Originally, the sketch and the written paragraphs appeared in separate pages. Due to space limitations they are presented here together.

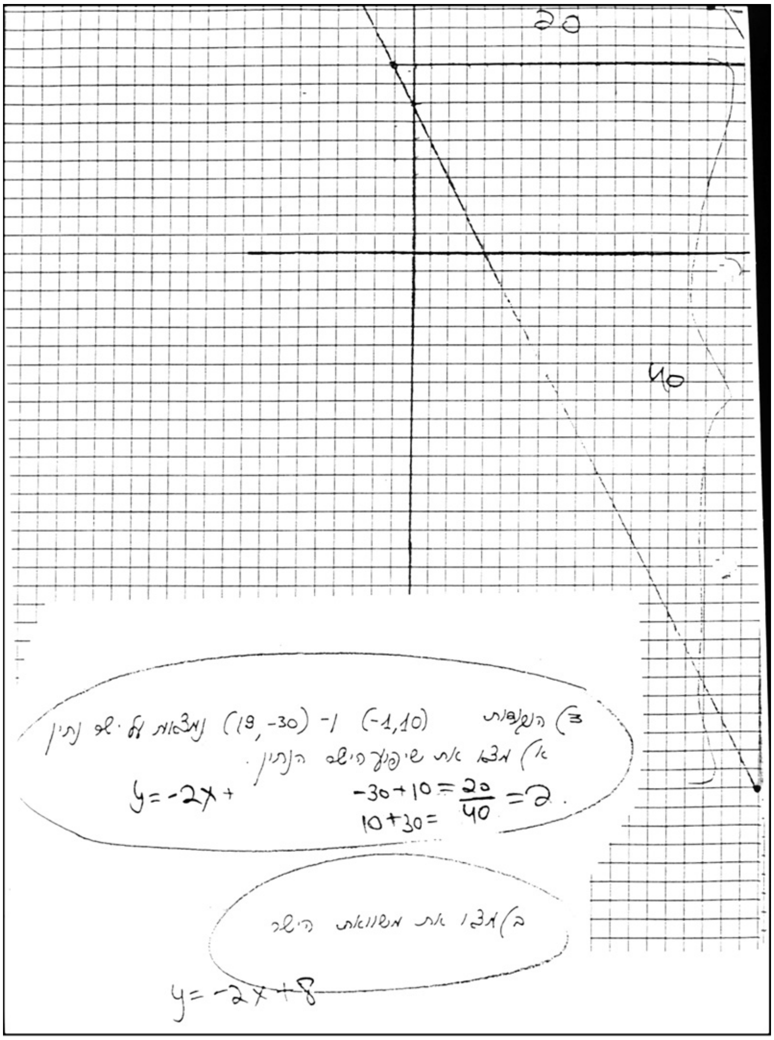


Fig. 4. A student's solution to a problem in analytic geometry (II).

of the line). However, instead of using the values of 40 and 20 directly from the graph, the impression is that the student tried to arrive at those values by "playing" with the numbers given in the question. Thus, the result of 20 was wrongly obtained from $-30+10$, while the result of 40 was obtained from $10+30$. Then, the quotient was found, again mistakenly, to be 2 (or shall we say, the quotient was "forced" to yield 2). These moves seem to be the outcome of a blur recollection of the slope formula ($m=(y_2-y_1)/(x_2-x_1)$), possibly combined with the 'rise over run' strategy.

What can we learn from this idiosyncratic calculation? The fact that, during an effort to write a short mathematical expression, the student made different errors (i.e., ignoring the x-values of the given points, taking the sum of the y-values instead of their difference, erroneous addition of a negative number and a positive number, and a mistake in division), is indicative of the difficulties that some low-track students face when they attempt to formalize their reasoning. However, in our view the important point is that besides the formal faulty product, there still exists an informal correct product, manifested in the final answer most likely obtained by graphical means. This product demonstrates the power of visualization as an informal tool, as it had enabled the student to handle the task when the more formal path was apparently full of obstacles.

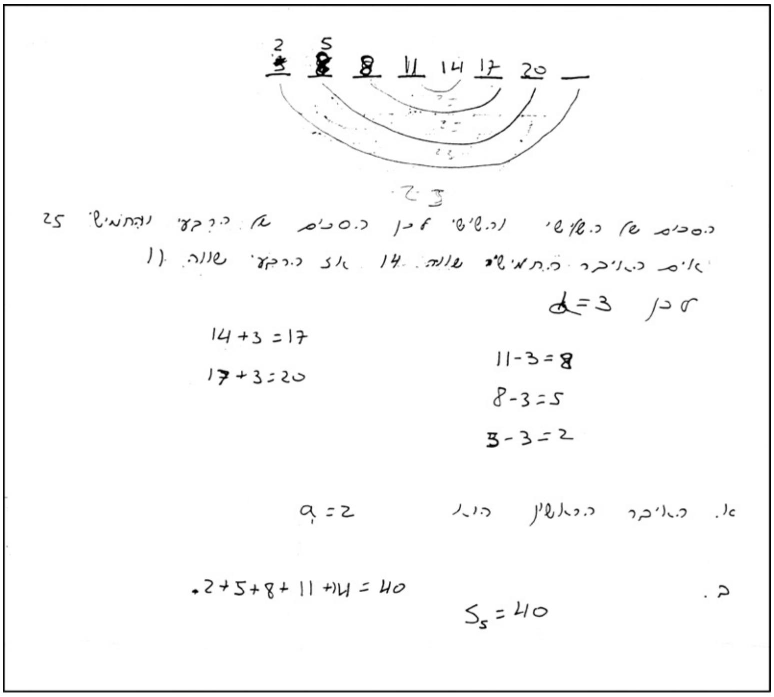


Fig. 5. A student's solution to an arithmetic sequence problem (II).

3.4. When informal products support self-monitoring

3.4.1. Example 4: self-correction of an error in a solution

This example refers to the mathematical topic of arithmetic sequences, as was the case in Example 2. However, this time the focus is on what seems, at first glance, to be a minor detail in the solution.

The question was:

In an arithmetic sequence, the sum of the third and sixth terms equals 25. The fifth term equals 14.

(a) Find the first term of this sequence.
(b) Find the sum of the first five terms of this sequence.

Fig. 5 shows a student's answer.

The Hebrew text written in the three lines following the diagram reads:

The sum of the third and the sixth and so the sum of the forth and the fifth 25 [sic].
If the fifth term equals 14 then the forth equals 11.
So $d = 3$.

Using the previously mentioned principle of arcs that connect pairs of symmetrically situated elements in the sequence, the student found a_4 to be 11 and then used this answer to obtain the correct value for the difference. Clearly, the student then used the difference of 3 backwards and forwards from the terms of 11 and 14, to calculate the missing terms. However, the reader may notice the following interesting detail: It seems that the student first erred in subtracting 3 from 11, as can be seen from the diagram. Our speculation is that 9 was obtained instead of 8, and that this error yielded the consequential mistakes of obtaining 6 and 3, instead of 5 and 2, as the second and first terms of the sequence (see the crossed over and corrected terms). How did the student discover this (or any possible other) mistake? From the relatively "clean" layout of the calculations appearing later in the answer, we speculate that the error was found and corrected during the previous step of filling the blank spaces by way of mental calculations. The repeated sum of 25 appearing (somewhat vaguely) under each arc suggests the possibility that the student checked the numbers to make sure they indeed satisfy this sum, and thus found the error (e.g. by realizing that $20+6 \neq 25$). This small detail is, in our opinion, quite significant. It could indicate that the student was aware that results must be coherent with one another, as well as with the given data. Such awareness should not be underestimated. In the mathematics education community, it is well recognized that developing the metacognitive skills of verification and evaluation of results is an important goal of mathematics learning at all levels (De Hoyos, Gray, & Simpson, 2004; NCTM, 1989, 2000). Garofalo and Lester (1985) have identified verification as one of four categories of desired metacognitive behaviors involved in solving a mathematical problem. Yet, many students tend to ignore or pay little attention to inspecting their final answers, once obtained (Pugalee, 2004; Stillman & Galbraith, 1998). When it comes to low achievers, the situation is worsened, due in part to insufficient direct metacognitive instruction (Cardlle-Elawar, 1995). In informal conversations we held with mathematics teachers of low-track students, many of them complained that their students seldom notice when they write down answers which make no sense, such as obtaining −4 for a side of a rectangle, or 1000 km/h for the speed of a car. This behavior corresponds with students' view of mathematics as an inscrutable subject, detached from their experience, which is handled by applying algorithms and procedures that have little to do with common sense.

Therefore, we can point to another advantage of informal strategies as used by low-track students. Grounded in common sense and intuition, such strategies yield results that are both attached to meanings and relatively easy to follow. When a contradiction appears, students are more alerted to detect it than they are when operating with algorithms, which are meaningless for them. We do not claim that the need to verify and check for errors comes naturally to students, but simply suggest that it is easier to encourage this metacognitive awareness when the mathematical context is one that makes sense to students. We argue, in accordance with the findings of Cardlle-Elawar (1995), that low-track students can, and should, learn to use self-monitoring tools and verification strategies, and that such tools should be integrated into the curriculum.

We conclude this subsection with a citation taken from an answer to a test item, written by another student in an experimental class of the 3U project. The student attempted to solve a problem in trigonometry, involving the calculation of the length of two altitudes and the area of a triangle. Due to an error at the very beginning, the values obtained for the two altitudes (calculated by the student using two separate triangle drawings) were incorrect. The student used both these values to calculate the area in two ways — each altitude multiplied by half of the corresponding side — and thus obtained two different results. Apparently unable to detect the root of the mistake, the student added the following lines:

> It doesn't make sense to me what I wrote in question 2, because for the altitude in triangle A I got 7.9 so the area is 31.6, and for the altitude in triangle B I got 9.6 so the area is 48, but the problem is that it is the same triangle with the same angles and the same sides, so the question is why did I get a different area?

The fact that the student tried to find the area in two ways can in itself be regarded as an existence proof to the claim made above. It suggests that the student had acquired at least one strategy of verification. But more than that, the student was highly aware of the contradiction that occurred as a result of using this strategy. Instead of crossing out the answer — as many students tend to do when they feel they have erred — it seems that this student felt the need to express a sense of dissatisfaction with the inconsistency discovered. Although the problem was not resolved, the student's written reflection can be considered a valuable learning outcome.

3.5. Daily life common sense at the service of informal mathematical products

3.5.1. Example 5: graph reading

Fig. 6 presents a graph describing the distance of a car from Tel-Aviv (in kilometers) as a function of the time (in hours).

Two questions are asked in regard to this graph:

(a) Between what hours did the car park?
(b) Between what hours was the car in its highest speed, and what was that speed?

This task was given as part of a classroom assignment to a low-track 10th grade mathematics class, studying according to the 3U curriculum (no graphing utilities were used). Students in this class were considered to be particularly low

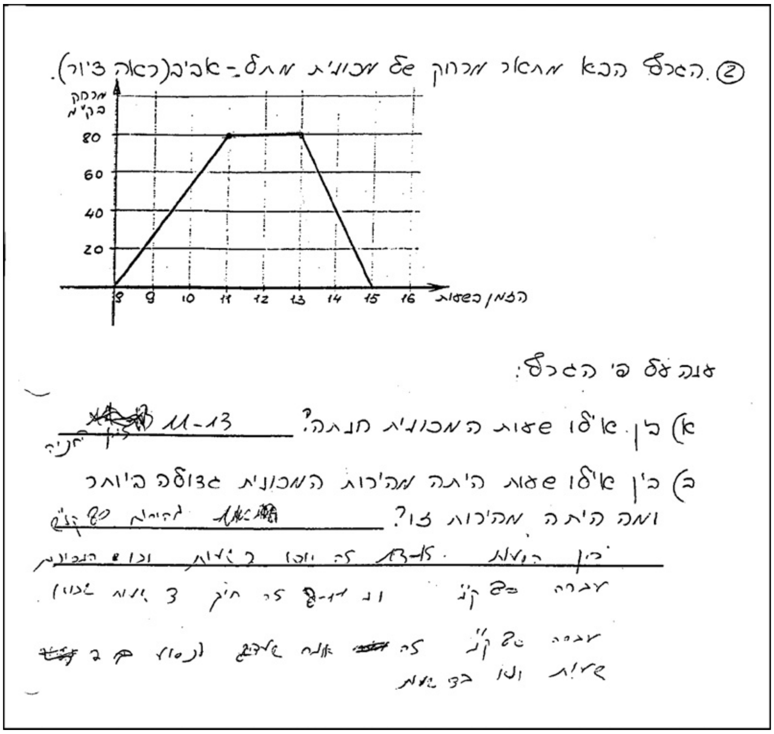


Fig. 6. A student's solution to a problem in graph reading.

in their mathematical achievements. In the lower part of Fig. 6, a student's answer is given. The student answered question (a) correctly (the answer reads: 11–13 parking time). However, question (b) posed a greater challenge as it requires ratio considerations, or, in other words, finding the absolute values of the slopes in different sections of the graph. The translated answer reads:

> Speed 80 km/h. Between the hours 13–15 it is 2 hours and in it the car passed 80 km. And from 8–11 it is in 3 hours that it passed 80 km. This means that it is better to drive in 2 hours and not in 3 hours.

The student begins the answer with an erroneous claim, apparently stemming from reading the value of 80, written on the vertical axis, as expressing velocity instead of distance. However, the rest of the answer is an interesting example of how an informal argument, which makes use of common sense and life experience, reveal that the student's understanding goes beyond an initial misinterpretation. The student relates to the two graph segments in which the car is shown to be moving (i.e., before and after the parking), and compares their features regarding time and distance. Clearly, the graph reading is now correct: the student identifies the value of 80 as kilometers traveled by the car (in both segments, regardless of the direction) and links this distance to the correct time intervals. Then, the student uses a practical consideration, taken from real-life experience ("it is better to drive in 2 hours and not in 3 hours") to indicate a higher speed, yet without a direct reference to the concept of speed, nor a calculation of the speed (although the answer evidently suggests that the relation between distance and time was taken into account).

We do not suggest that this informal product is sufficient. Indeed, the student did not find the highest speed, and there is no evidence that he knew how to calculate speeds. Our basic premise is that there is potential in such an answer. It indicates once again that students are capable of sound reasoning in situations grounded in a familiar context. We claim that it is possible to recruit these competencies in order to strengthen mathematical understandings of low achievers. In this particular case, for instance, a valuable course of action on the part of a tutor could be reinforcing the contemplation of "better to pass the same distance in less time", and a gradual transition towards the mathematical operation of dividing distance by time in order to obtain a measurement for speed. The process of reinforcing existing ideas and treating them as sprouts of more developed (and mathematically entrenched) notions, is well-known in mathematics and science education (see, for example, Smith, diSessa & Roschelle, 1993). However, implementing such processes with low-achieving secondary school students sets this issue in a unique context that so far has been discussed by few researchers (one exceptional work is that of Chazan, 2000). In our last example below, we demonstrate such a process. This concluding case serves as grounds for looking into important elements associated with teaching low achievers in mathematics.

3.6. Correctness vs. productiveness of informal mathematical products

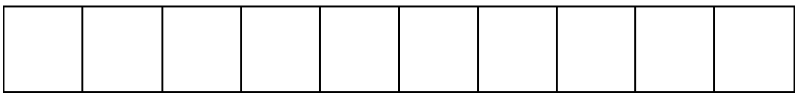
3.6.1. Example 6: Dan explores the decimal representations of 3/4 and 1/4

Dan was a ninth grade student when interviewed as part of the Middle-School Classroom Study on Low-Track Mathematics. He was studying mathematics in the lowest track offered in his high school, and was described by his mathematics teacher as a “typical student” of this low-track group, having average grades comparing to his peers. Dan’s mathematics class was observed during the time when the concept of functions was introduced and explored. We invited Dan for an interview out of class, conducted by the first author (RK). The interview’s goal was to compare Dan’s formal and informal knowledge about functions, but in the course of the interview some of Dan’s interesting ideas about decimal fractions were revealed. The part reported here refers to these ideas.

Dan had difficulties in determining what 3/4 would be when written as a decimal. He hesitantly conjectured that it would be 0.7, but when offered a calculator to check this conjecture, he said[11]: “I don’t know how to do this. I don’t know how to convert this. No, I don’t think I can”. Instead, he offered the following explanation to show that 3/4 is 0.7:

> “In order to make a whole I need to make me [sic] four quarters. Four quarters is four and four, it’s eight, [. . .] three quarters so it’s almost a whole, one quarter is missing, so it’s seven. If we say one more quarter it will make eight which is the whole.”

Continuing this line of thought, Dan further conjectured that *one* quarter is 0.1. It therefore seemed that the correct formal procedure of converting fractions into decimals was not available to Dan, and converting was instead carried out by idiosyncratic intuitions. Nevertheless, Dan did remember (apparently due to a more frequent use) that one half is 0.5. When asked to explain this fact, he claimed that here the whole is 10, and half of it is 5. At this point RK introduced a spontaneous intervention, explaining to Dan that in the decimal system the whole is always 10. She presented him with a simple diagram of a ten-squared rectangle, basically looking like the one below:



RK emphasized that one-half is indeed 0.5, since half of this whole rectangle is made of five squares. Dan was then requested to use the same idea for determining what will one quarter be when written as a decimal. The conversation which evolved from this point onwards, cited below, reveals Dan’s attempt to make use of this visual tool and reach the answer, through a series of trial-and-error speculations about the number of squares that will “fit in” the whole exactly four times.

Dan: Ah . . . It will be . . . [marks division lines after every two squares]. Each two squares is 0.2 [sic]. So four times . . . ah . . . five times 0.2 is ten.
RK: That is, the whole.
Dan: The whole. So a quarter is 0.2.
RK: How many times does a quarter go into the whole?
Dan: Five times.
RK: And this makes sense to you?
Dan: No.
RK: Why not? How many times should it go in?
Dan: Four.

[11] The interview was recorded and transcribed. Citations are translated from Hebrew.

RK: So maybe you can figure out what to do in order to better the situation, so it will go in only four times and not five times?
Dan: Ah . . . [pause, Dan seems to be concentrating]
RK: We can make a new drawing [draws a similar rectangle, Dan scribbles lines on it].
RK: What are you trying?
Dan: I tried three three three, it's not.
RK: Why not?
Dan: Three times three is nine, so I'm missing one more three to get four quarters.
RK: So what do you figure out of this?
Dan: That it's not three.
RK: On the other hand, we also saw that it's not two.
Dan: So it's one and a half.
RK: Do you want to mark every one and a half and see how many times it goes in?
Dan: Wait a minute, it's two and a half, because if it's one and a half then it will be larger than the five, about six.
RK: And the two and a half, will you show me in the drawing? [Dan marks groups of two and a half squares]. What do you think of this answer?
Dan: That it's true.
RK: You sound certain, how do you know?
Dan: Because there is one, two, three, four here.
RK: So what can be concluded about a quarter if we want to write it as a decimal?
Dan: A quarter is zero point two and a half.
RK: So how do you write it?
[Dan writes 0.2 1/2]
RK: Is there another possibility maybe?
Dan: I think, but I'm not sure, that you can make the dot twice. 0.2.5.

Dan's case portrays a possibility to provide instruction that will help creating a "bridge" between informal intuitions and more formalistic representations of mathematical ideas, relying on common sense and minimal existing knowledge. Clearly, in the beginning of the interview section cited above and throughout the conversation, Dan had not recalled any rote-learned information about the formal system of decimals, except for the fact that a half is represented by 0.5. By the end of the section he had "invented", in the sense offered by diSessa et al. (1991), his own formal representation — or rather, two representations — of 1/4 as a decimal.

We suggest that these representations were based on a "cognitive bridge", stretching between Dan's initial informal and idiosyncratic conceptions of part-whole relationships as a basis for determining the decimal form, and the formal, conventional way of deriving decimals from part-whole ratios in the base-10 system. Through the interaction, Dan was provided with an opportunity to become actively involved in appropriating (as discussed by Moschkovich, 2004) new meanings and representations for the concept he had about decimals. Although the representations offered by Dan (0.2 1/2, 0.2.5) are different from the one accepted by the mathematical community, they are nevertheless legitimate representations, since they are consistent with the underlying concepts and ideas of decimals.[12] The main point is that Dan, in spite of an apparent deficiency in the relevant arithmetic basis, has "created" a formal mathematical representation, through an interaction that exploited his common sense and some preliminary ideas about decimals. While most of these ideas may well be regarded as incorrect, it is possible to look upon them as *productive*, in the sense introduced by Smith et al. (1993), who claimed that "It is impossible to separate students' misconceptions, one by one, from the novice knowledge involved in expert reasoning" (p. 147), and therefore suggested that "Productive or unproductive is a more appropriate criterion than right or wrong" (p. 147). Dan's first explanation (justifying why 1/4 is 0.7) seems to reflect a misconception held or constructed ad-hoc by Dan in the absence of learned tools.

[12] An examination of the history of mathematics shows that representations similar to that of Dan's have appeared during early stages of the development of written decimals. According to Smith (1925, republished 1958), Pitiscus occasionally used several decimal points for a single number, as appears in his tables published in 1612. Also, as Smith notes, "even as late as 1655 we find the period used to separate an integer and common fraction, as in the case of 198. 1/2" (p. 246). In our modern times we also find some resemblances to Dan's writing of 0.2.5: For example, in Israel's verbal report of exchange rates, numbers such as 4.367 are read as "four Shekels, 36 cents ("Agorot") and seven tenth of a cent".

According to this misconception, the "whole" in the part-whole relationship used to determine the decimal form of a fraction, depends on the denominator of that fraction. The intervention introduced to Dan was built on the "kernel of correctness" underlying this misconception, i.e., that part-whole ratios are indeed a key to converting fractions into decimals. However, it stressed that the "whole" in the system of decimals is always 10 — an idea that connected well to Dan's existing understanding of why a half is represented by 0.5.

The notions of "productivity" and "kernel of correctness" can be considered as key elements in the process of evaluating mathematical products of all learners, but even more so when it comes to low achievers. Chazan (2000) claims that mathematics teachers tend to emphasize correctness, in its most rigorous meaning, as an exclusive criterion, and are thus prone to miss important opportunities to advance weak students. Many students, being frequently unable to reach the precise answer expected by the teacher, withdraw from sharing their ideas with him (or her). This practice not only decreases the students' chances to benefit from helpful instruction, but is also harmful in terms of affective results. The following citation, taken from a study investigating adults' retrospection on high school mathematics (Karsenty, 2001, 2004), illustrates this point. Ilan, a 40-year-old businessman, describes the teacher who taught his mathematics class more than 25 years ago. The description highlights a class atmosphere created by the need to produce correct answers:

> "He was an authoritarian teacher. He would make you flounder [. . .]. He would, let's say, ask Jimmy a question.[13] Jimmy doesn't know. Well, he turns to Jimmy 2, Jimmy 3, picking up his few Jimmys every lesson, asking them a question. If they answer correctly he says "good", if they answer wrong, he nods 'wrong' with his head. Then he would begin to develop formulas, asking who understands and who doesn't, of course no one dared to raise a hand, including those who did understand, because then he'll ask them, and those who didn't understand, because he'll ask them even more. [. . .] I studied mathematics very technically, and either I have reached the correct answer or I haven't. If I didn't reach the correct answer, I received a negative feedback, if I reached the correct answer I received a positive feedback and that's it. This is how I learned mathematics. I had to master the technique and reach the correct answer. In my opinion, it was not right for me and not right in general. It's missing the target. I don't think I'm projecting from myself to everyone, but I'm one of millions of examples of how one should not learn mathematics." (Karsenty, 2001, translation from pp. 285–286).

Although this quote relates to lessons that took place during the seventies, and does not refer to a low-track class, we suggest that the affective imprint it conveys may be similar for many classes today, where mathematics is taught through methods still rigidly prioritizing the right/wrong dichotomy. For an average student, such as Ilan (as he described himself elsewhere), the effect of such teaching might be manifested in a long-term feeling of mistreatment. For a low-achieving student, the same circumstances may be far more detrimental. If there is no room in the mathematics class for intuitive half-baked and perhaps half-correct ideas, as springboards for learning, then students who cannot present correct formal arguments or answers will at best keep quiet, and in other cases will negatively "contribute" to the lesson.

To point to a different possibility, we offer an example of a mathematical product created in class by a student. The atmosphere in this class, observed as part of the 3U Project, was quite the opposite from the one described above by Ilan. Students spoke their minds freely — sometimes maybe too freely — and almost all comments were addressed, in one way or another, by the teacher. During the year, students have learned extensively about linear graphs and their algebraic representations as equations of the form $y = ax + b$.[14] Towards the end of the year, the teacher turned to introducing the subject of arithmetic sequences. She gradually led the students to construct a way for finding the *n*th term in a sequence, later arriving at the written expression of $a_1 + (n - 1)d$. It was then that Noam, a student with a record of very low achievements in mathematics, had said:

> "The a_1 is the starting point, like b is the starting point of a straight line, and d is the slope."

If we are to judge Noam's remark by the 'pure correctness' criterion, we may have to reject the analogy he offered, since it contradicts a basic mathematical fact: a straight line represented by $y = ax + b$ has no "starting point". However, if we adopt a more open-minded view on this analogy, we should be quite impressed with the student's ability to see

[13] In the original interview text, Ilan used a Hebrew common name, which we changed here to "Jimmy".

[14] Earlier we referred to linear equations as represented by $y = mx + n$. However, in many Israeli classes teachers use the parameters *a* and *b* instead of *m* and *n*.

linearity in arithmetic sequences — a concept that has only lately been introduced to him — and furthermore, his ability to specifically identify d as the "slope" of the sequence. This is hardly an observation that teachers often come across with in low-track mathematics classes (moreover, it is likely that many students in higher tracks do not connect between the concepts of arithmetic series and linearity). Indeed, Noam was complimented for this comment by the teacher. As for the incorrect term of a "starting point", the teacher was well aware of what Noam referred to: When drawing a linear graph, her students usually marked the y-intercept point of the line, which is determined by the value of b, and from it continued the line in both directions, according to steps determined by the value of a. So, what Noam saw as a "starting point", was not a mathematical formal notion, but rather an intuitive, operational term related to drawing the line. In addition, one can think of mathematical situations in which Noam's notion of a starting point will in fact be correct: various physical occurrences, for instance, are described for non-negative variables only. When outcomes of such occurrences are modeled by linear graphs, it makes perfect sense to consider the starting point (or, in other words, the initial condition) of the straight line. Noam's intuition has thus a clear "kernel of correctness" in it. However, this important point might have been overlooked by a teacher who is determined to "hear what s/he expects to hear". Moreover, the supportive approach of the teacher in this case, which made it possible for an informal mathematical product to be articulated out loud and appreciated, is unfortunately not to be taken for granted. In a discussion held with teachers as part of the SHLAV Project, teachers were presented with an example taken from Chazan (2000), in which a student has expressed the function $y = x^2$ as $y = x{\cdot}x + 0$, and concluded from this "linear" format that the slope of x^2 was x. Hearing such a suggestion, the "kernel of correctness" that could be picked up by an attentive teacher (even if the student was unaware of it) is the notion of the slope as *changing*, instead of its constant form in linear functions. In this sense the student's suggestion certainly satisfies the criterion of productiveness (Smith et al., 1993). However, some of the teachers could not accept this idea. They heartily claimed that it is a fundamental mistake to present x^2 as linear, and that since it is the responsibility of mathematics teachers to ensure the accuracy of mathematical information presented in class, they must therefore reject such answers altogether. Needless to say, such an attitude on the part of teachers is a critical element when it comes to low achievers, in light of all that has been discussed so far in this article.

4. Summary

We have defined and explored the notion of informal mathematical products of secondary school low achievers. Through the illustrative examples presented, several features of informal products were described and discussed. Firstly, we have related to the fragility of products, concerning situations when sound mental reasoning disintegrates during attempts to transform it into a written formal language, or when the mental product is not solid enough to be articulated as a convincing argument. The need to provide low achieving students with writing tools and skills, which may enable them to both communicate and reinforce their informal reasoning, was raised. Secondly, we have examined how informal mathematical products are built upon resources available to students, such as common sense, life experience and visual understanding. The examples demonstrated how students manage to solve tasks, while bypassing formulae and rigid symbolic language. Thirdly, it was pointed out that as a result of being grounded in students' common sense, informal mathematical products are likely to integrate some metacognitive skills such as self-monitoring, verification and awareness to the reasonableness of answers. Lastly, we have discussed the potential of informal mathematical products as springboards to knowledge growth. Stressing the productiveness of students' informal reasoning, instead of judging it purely against the formal correctness criterion, can serve as an instructional approach which is significantly different from traditional approaches.

Taking a holistic view at the elements mentioned above, we see them as converging to an important point. Simply put, the point is that many students who experience great difficulties in mathematics are nevertheless capable of generating productive mathematical ideas. Their thinking products may be expressed in non-formal or non-conventional ways, may include inaccuracies or reflect aspects that are traditionally excluded from the common mathematical discourse, yet they are valuable and should be treated and nurtured as such. Beyond the discussion of how far mathematical literacy should go (or how much formal math should *all* students learn), we must acknowledge the importance of informal mathematical products within any mathematics classroom, legitimize and harness them in order to exploit the reasoning capabilities that many students possess. Metaphorically speaking, teaching in that respect is aimed at encouraging students to pull themselves up by their own bootstraps, because it is their own productions that serve as starting points for learning. Through informal methods, students can successfully handle required mathematical tasks, which might have been inaccessible if only traditional formal strategies were to be used.

Notwithstanding all the above, it is also important to recognize the limitations of informal approaches. Different topics, within secondary school mathematics, vary in the degree to which they lend themselves well to informal presentation that does justice with the expected level to be reached. By no means do we suggest dismissing formal aspects of mathematics and concentrating solely on those aspects that allow for non-symbolic informal treatments. On the contrary; it is important to engage students in interesting tasks, for which the solution involves some formalism. Also, one cannot ignore the fact that ultimately students will have to confront some formalism in order to complete their mathematics high school education, and from a social perspective it is crucial that they get a fair chance to handle more formal tasks. Yet, it is *the road* taken on the way to formality that concerns us here, rather than the final target. Our claim is that rushing into formal mathematical outcomes, without taking into consideration the intuitions and informal ideas of students, might weaken potential strengths of learners, especially low achievers. The process of acquiring mathematical concepts should be based upon *continuity* (Smith et al., 1993), that is, taking existing informal knowledge as a starting point, refining and extending it through activities of "learning by doing", while aiming to eventually solidify it at a more formal level. We have related to opportunities to apply such "bridging" in a couple of examples presented in this paper, especially in the case of Dan, where a process of appropriation was described. As already stressed, the issue of how the theoretical ideas of continuity, bridging and appropriation may be enacted in low-track settings is an issue that calls for further research. Questions such as how, and under what terms, can formality be based on informality to produce meaningful learning by secondary school low-achievers are still insufficiently addressed by research, neither from an academic-theoretical nor from a practical perspective. Further investigations should be dedicated, among other questions, to the following:

- Which topics, included in secondary school mathematics, carry intrinsic possibilities for a successful bridging between the formal and informal ideas embedded in them? In contrast, within which topics will informal mathematical products be of a more limited power, due to considerable ruptures between formal and informal reasoning?
- How can mathematics teachers become more aware of the learning potential of informal mathematical products created by low achievers?
- What would be the characteristics of useful teaching models that rely on these products as significant resources? How can central teacher activities, such as design of classroom tasks and assessment of students' work, which in many low-track settings are still of a dreary and archaic nature, be revitalized in ways that will promote students' participation and depict their understanding in a reliable manner? How can teachers motivate low-track students to develop their informal products into more formal ones?

Several steps may be suggested as means for tackling these questions:

Documenting teaching episodes involving low-achievers (either in low-track classes or in more heterogeneous settings), in order to analyze teachers' behaviors which contribute to the legitimization of informal products; Examining various conditions (regarding, for instance, school policies, curriculum preferences, physical environments, etc.) that enable the evolvement of a class atmosphere in which low-achievers can articulate their informal ideas; Designing intervention plans for pre-service and in-service teachers on this issue, and assessing their effectiveness. Another avenue of research relates to the analysis of continuity as opposed to unbridgeable ruptures within various mathematical topics and, in accordance with such analysis, design of experimental instructional approaches. These, and more, can be viewed as necessary bricks on the road leading to equity in mathematics education.

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